Graph Detection Theory for Power Law Graphs

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Outline

- **Introduction**
  - ***Goals***
  - ***Detection Theory***
  - ***Sparse Matrix Duality***
- **Backgrounds and foregrounds**
- **Tree Finding**
- **Summary**

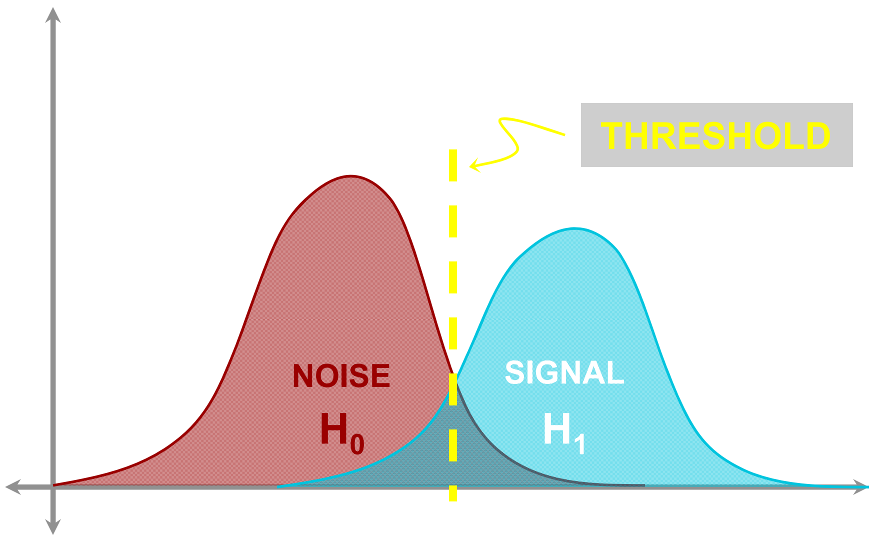
Goals

- **Detection Theory**
  - Apply basic postulates of detection theory (signal, background, …)
  - Quantitatively estimate difficulty of problem (SNR)
  - Develop better detection algorithms

- **Linear Algebraic Graph algorithms**
  - Additional tools for algorithm development
  - Compact representation
  - Parallel implementation well understood

Detection Theory

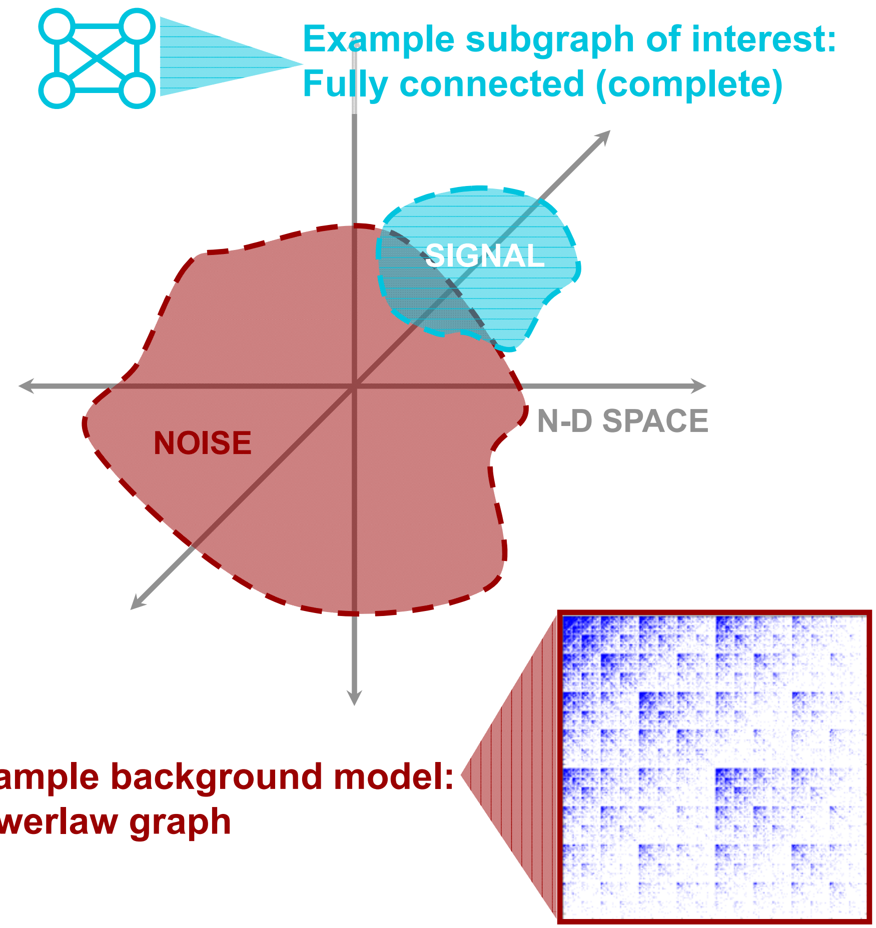
DETECTION OF SIGNAL IN NOISE



ASSUMPTIONS

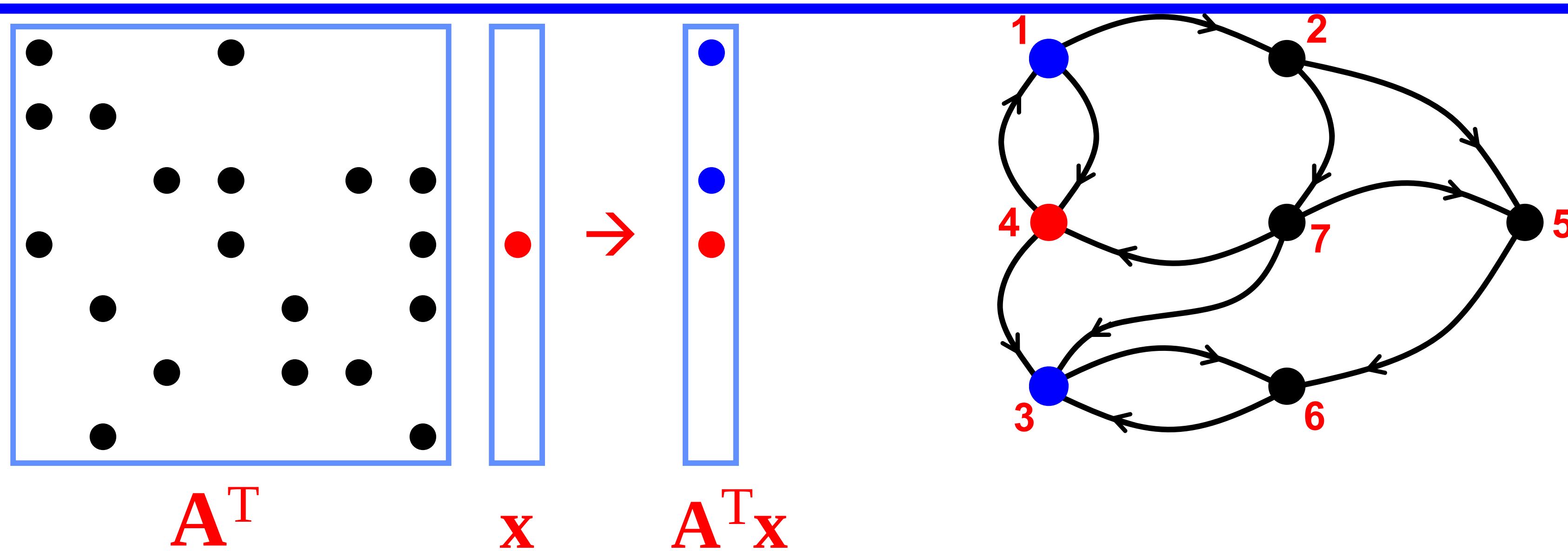
- Background (noise) statistics
- Foreground (signal) statistics
- Foreground/background separation
- Model ≈ reality

DETECTION OF SUBGRAPHS IN GRAPHS



Goal: Develop basic detection theory for finding subgraphs of interest in large background graphs

Graphs as Matrices



- Graphs can be represented as a sparse matrices
  - Multiply by adjacency matrix → step to neighbor vertices
  - Work-efficient implementation from sparse data structures
- Most algorithms reduce to products on semi-rings: C = A "+"."x" B
  - "x" : associative, distributes over "+"
  - "+" : associative, commutative
  - Examples: +.* min.+ or.and

Algorithm Comparison

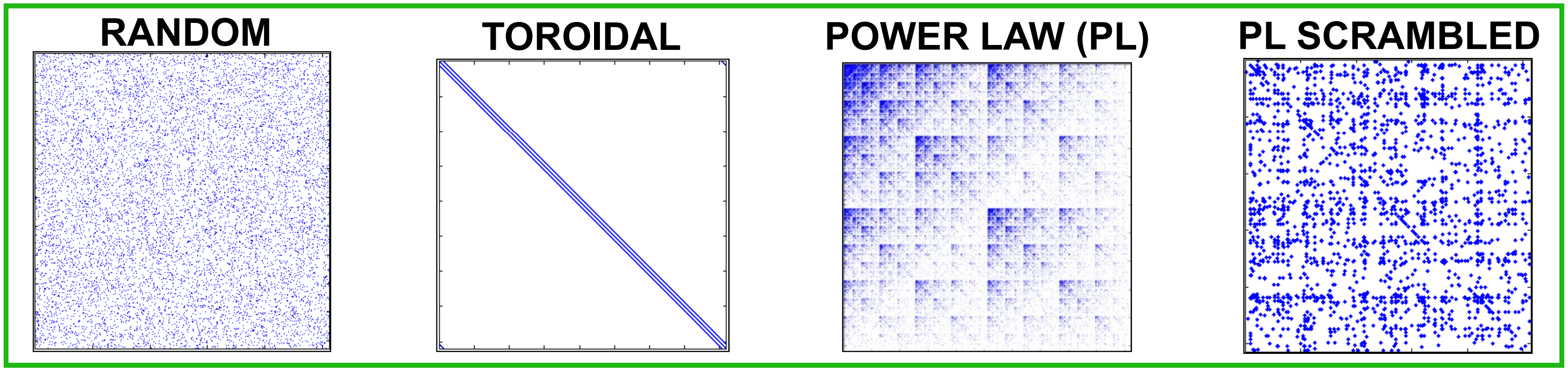
Algorithm (Problem)	Canonical Complexity	Array-Based Complexity	Critical Path (for array)
Bellman-Ford (SSSP)	$\Theta(mn)$	$\Theta(mn)$	$\Theta(n)$
Generalized B-F (APSP)	NA	$\Theta(n^3 \log n)$	$\Theta(\log n)$
Floyd-Warshall (APSP)	$\Theta(n^3)$	$\Theta(n^3)$	$\Theta(n)$
Prim (MST)	$\Theta(m+n \log n)$	$\Theta(n^2)$	$\Theta(n)$
Borůvka (MST)	$\Theta(m \log n)$	$\Theta(m \log n)$	$\Theta(\log^2 n)$
Edmonds-Karp (Max Flow)	$\Theta(m^2 n)$	$\Theta(m^2 n)$	$\Theta(mn)$
Push-Relabel (Max Flow)	$\Theta(mn^2)$ (or $\Theta(n^3)$)	$O(mn^2)$	?
Greedy MIS (MIS)	$\Theta(m+n \log n)$	$\Theta(mn+n^2)$	$\Theta(n)$
Luby (MIS)	$\Theta(m+n \log n)$	$\Theta(m \log n)$	$\Theta(\log n)$

Majority of selected algorithms can be represented with array-based constructs with equivalent complexity.

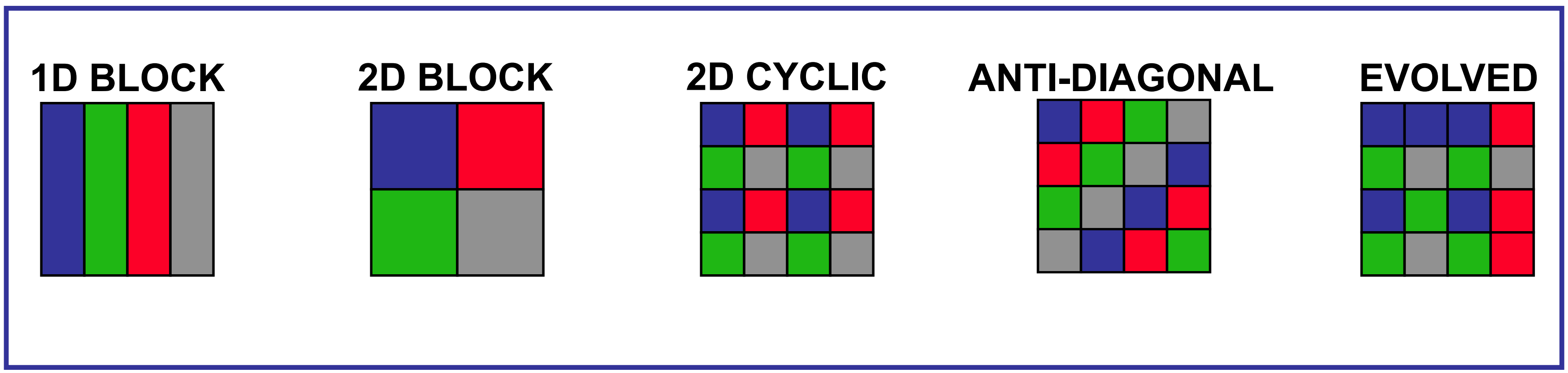
($n = |V|$ and $m = |E|$.)

Distributed Array Mapping

Adjacency Matrix Types:



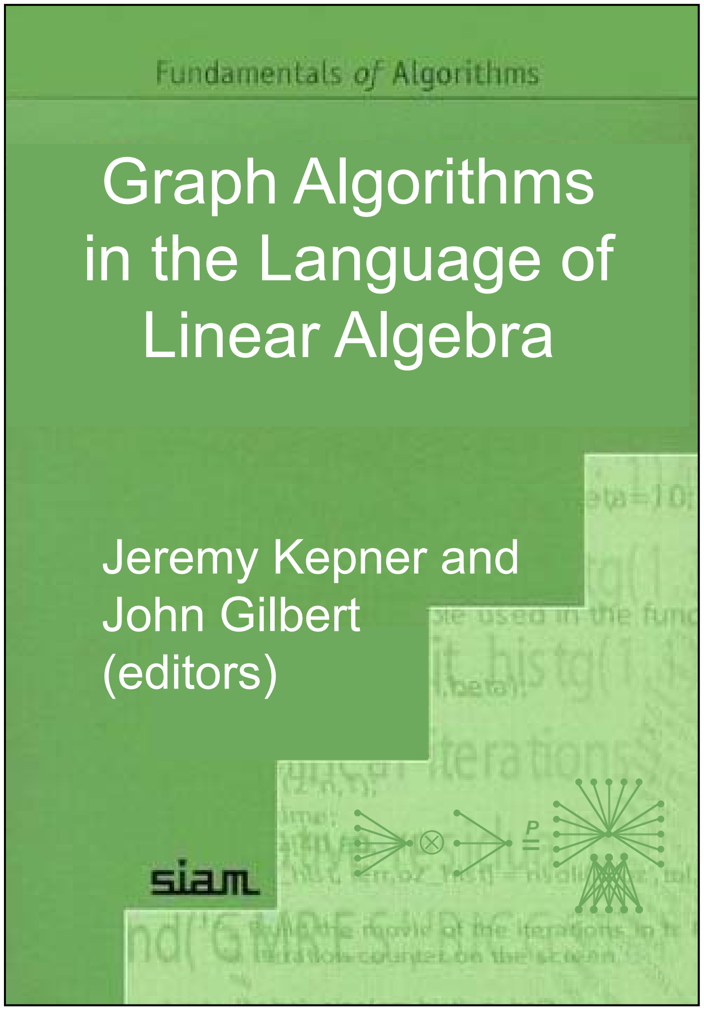
Distributions:



Sparse Matrix duality provides a natural way of exploiting distributed data distributions

Reference

- **Book: "Graph Algorithms in the Language of Linear Algebra"**
- **Editors: Kepner (MIT-LL) and Gilbert (UCSB)**
- **Contributors**
  - **Bader (Ga Tech)**
  - **Chakrabart (CMU)**
  - **Dunlavy (Sandia)**
  - **Faloutsos (CMU)**
  - **Fineman (MIT-LL & MIT)**
  - **Gilbert (UCSB)**
  - **Kahn (MIT-LL & Brown)**
  - **Kegelmeyer (Sandia)**
  - **Kepner (MIT-LL)**
  - **Kleinberg (Cornell)**
  - **Kolda (Sandia)**
  - **Leskovec (CMU)**
  - **Madduri (Ga Tech)**
  - **Robinson (MIT-LL & NEU), Shah (UCSB)**

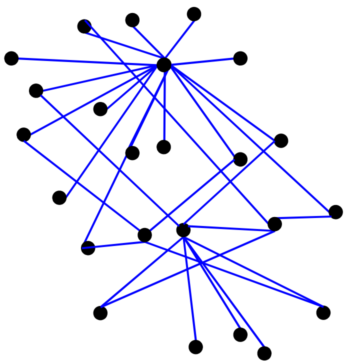


Outline

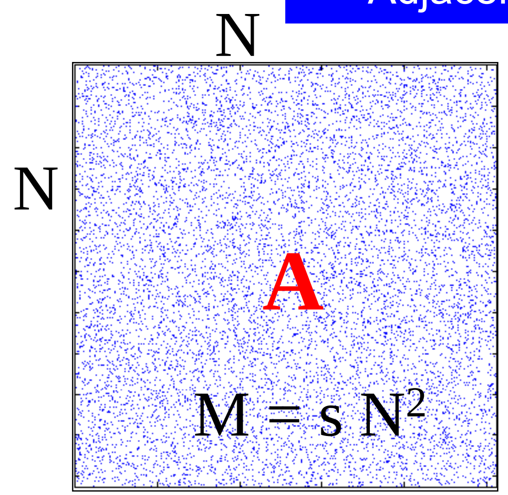
- Introduction
- **Background and foregrounds**
  - *Random*
  - *Power Law*
  - *Clique*
  - *Source/Sink*
  - *Tree*
- Tree Finding
- Summary

Background: Random (Erdos-Renyi)

Graph



Adjacency Matrix



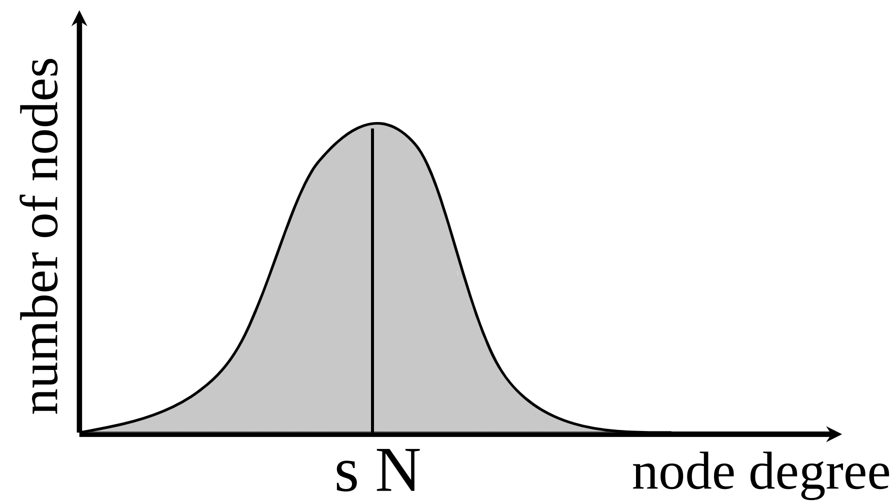
Algebraic Form

$$\mathbf{A} : \mathbb{B}^{N \times N}$$

$$\mathbf{A}(i,j) = (r < s)$$

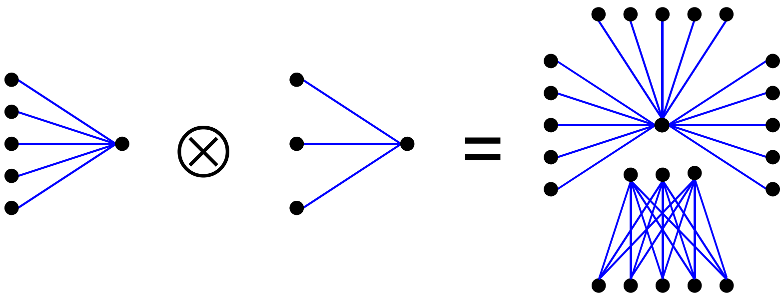
$$r \leftarrow [0,1]$$

Degree Distribution

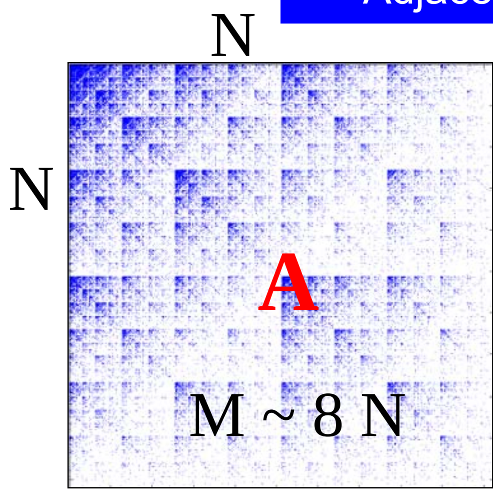


Background: Power Law (Kronecker)

Graph



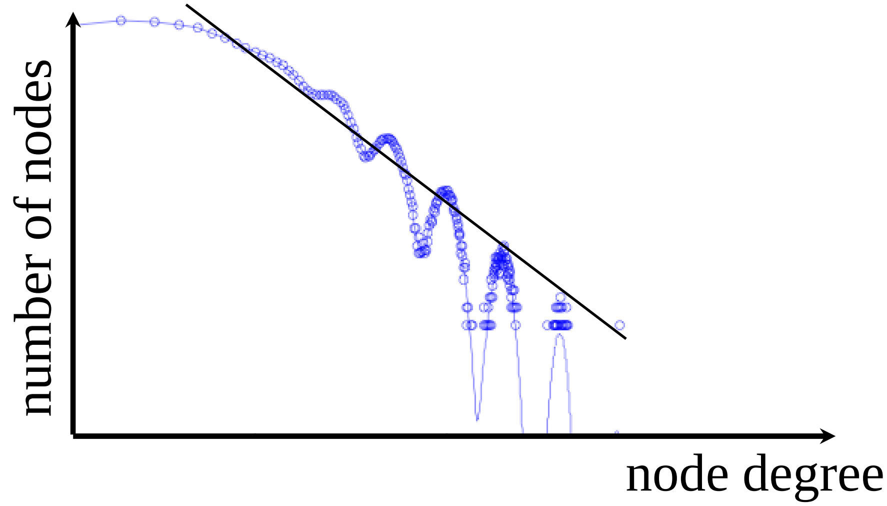
Adjacency Matrix



$$\mathbf{G} : \mathbf{R}^{n \times n}$$

$$\mathbf{A} \overset{M}{\leftarrow} \mathbf{G}^{\otimes k} = \mathbf{G}^{\otimes k-1} \otimes \mathbf{G}$$

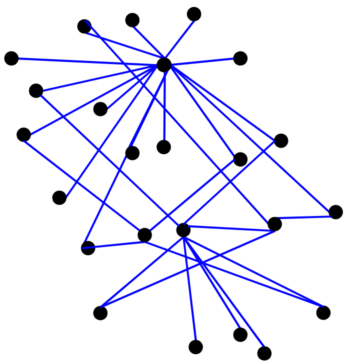
Algebraic Form



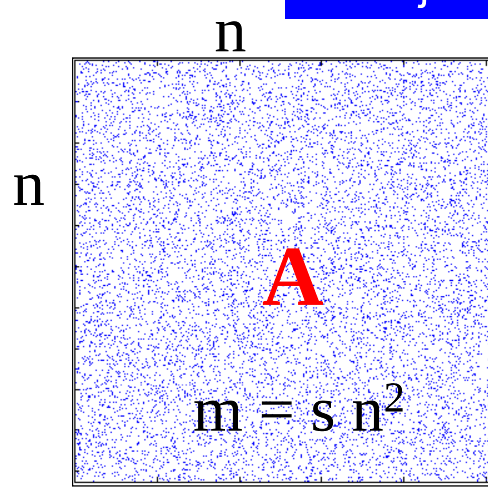
Degree Distribution

Foreground: Clique (Partial)

Graph



Adjacency Matrix



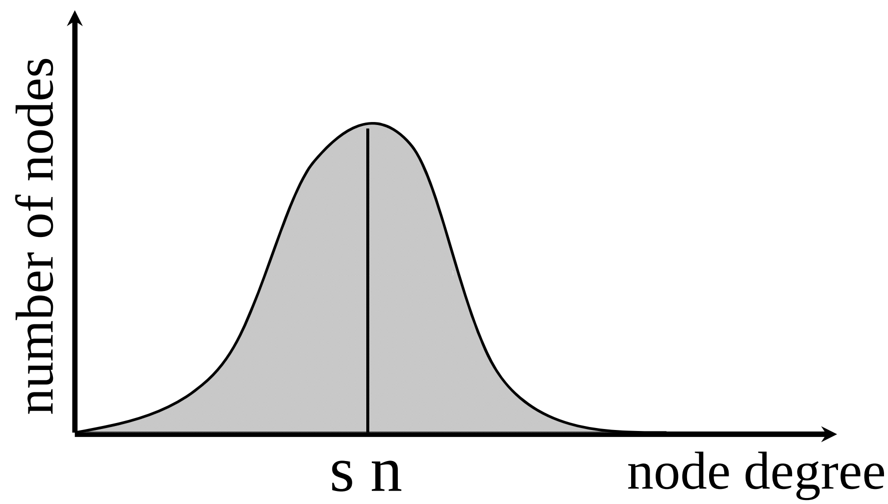
Algebraic Form

$$\mathbf{A} : \mathbb{B}^{n \times n}$$

$$\mathbf{A}(i,j) = (r < s)$$

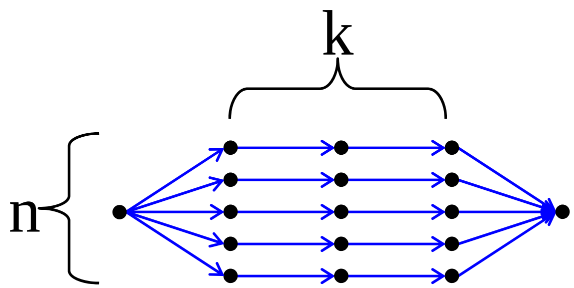
$$r \leftarrow [0,1]$$

Degree Distribution

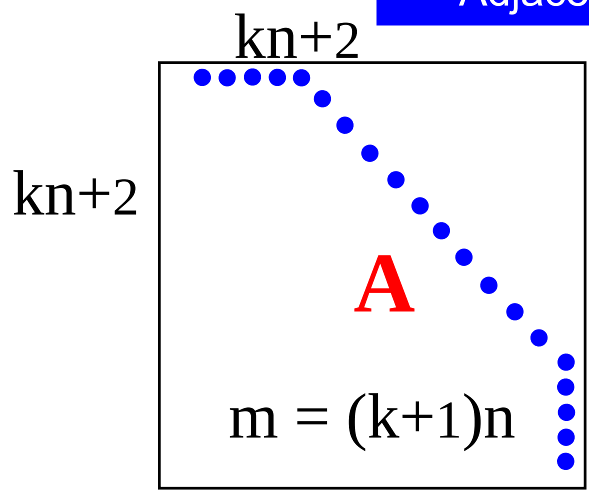


Foreground: Source Sink

Graph



Adjacency Matrix

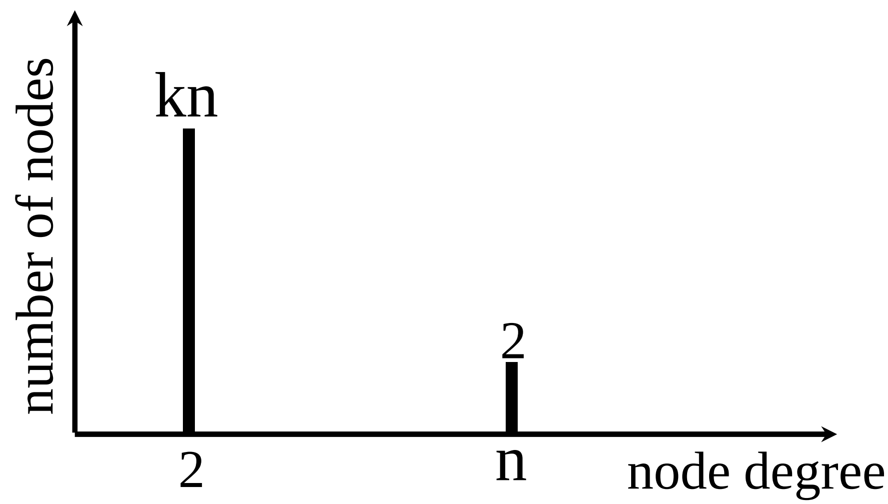


Algebraic Form

$$\mathbf{A} = \overset{k\times k}{\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0\end{pmatrix}} \otimes \overset{n\times n}{\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}} +$$

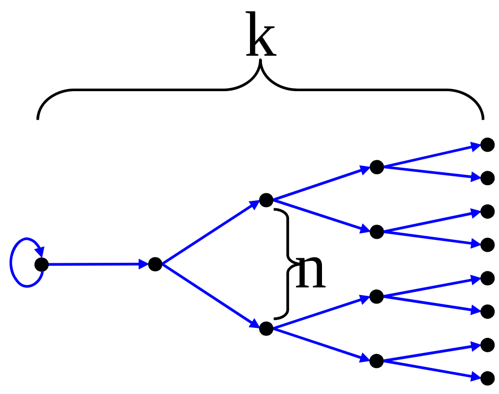
$$\underset{k\times k}{\begin{pmatrix}1 & & \\ & & \\ & & \end{pmatrix}} \otimes \underset{1\times n}{\begin{pmatrix}1\end{pmatrix}} \otimes \underset{n\times 1}{\begin{pmatrix}1 \\ \vdots \\ 1\end{pmatrix}'} + \underset{k\times k}{\begin{pmatrix} & & \\ & & \\ & & 1\end{pmatrix}} \otimes \underset{n\times 1}{\begin{pmatrix}1\end{pmatrix}'} \otimes \underset{1\times n}{\begin{pmatrix}1 \\ \vdots \\ 1\end{pmatrix}}$$

Degree Distribution

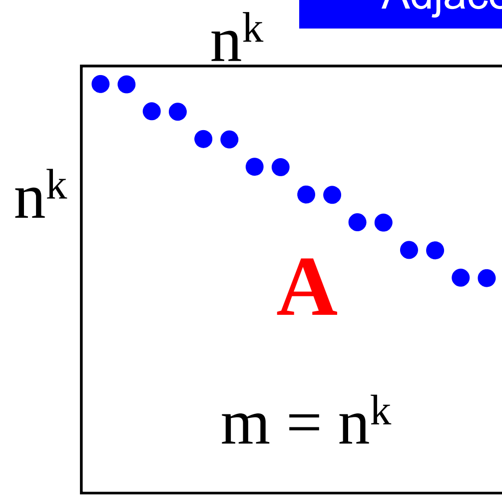


Foreground: Trees

Graph



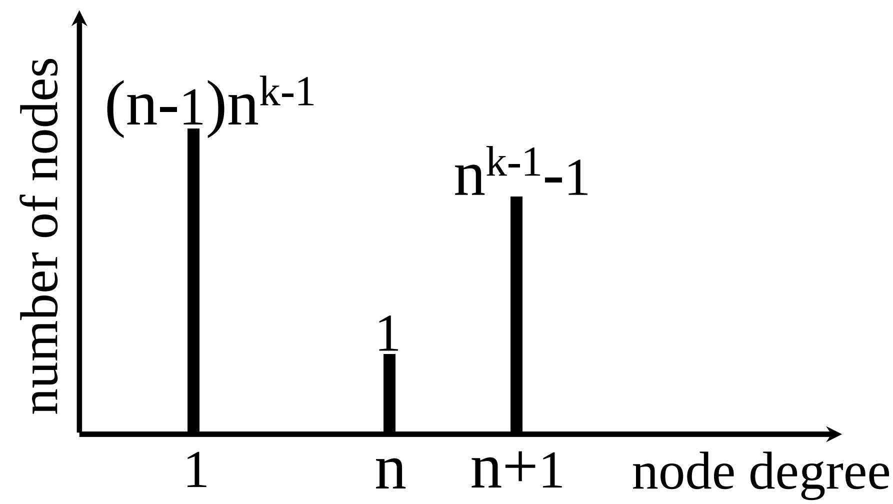
Adjacency Matrix



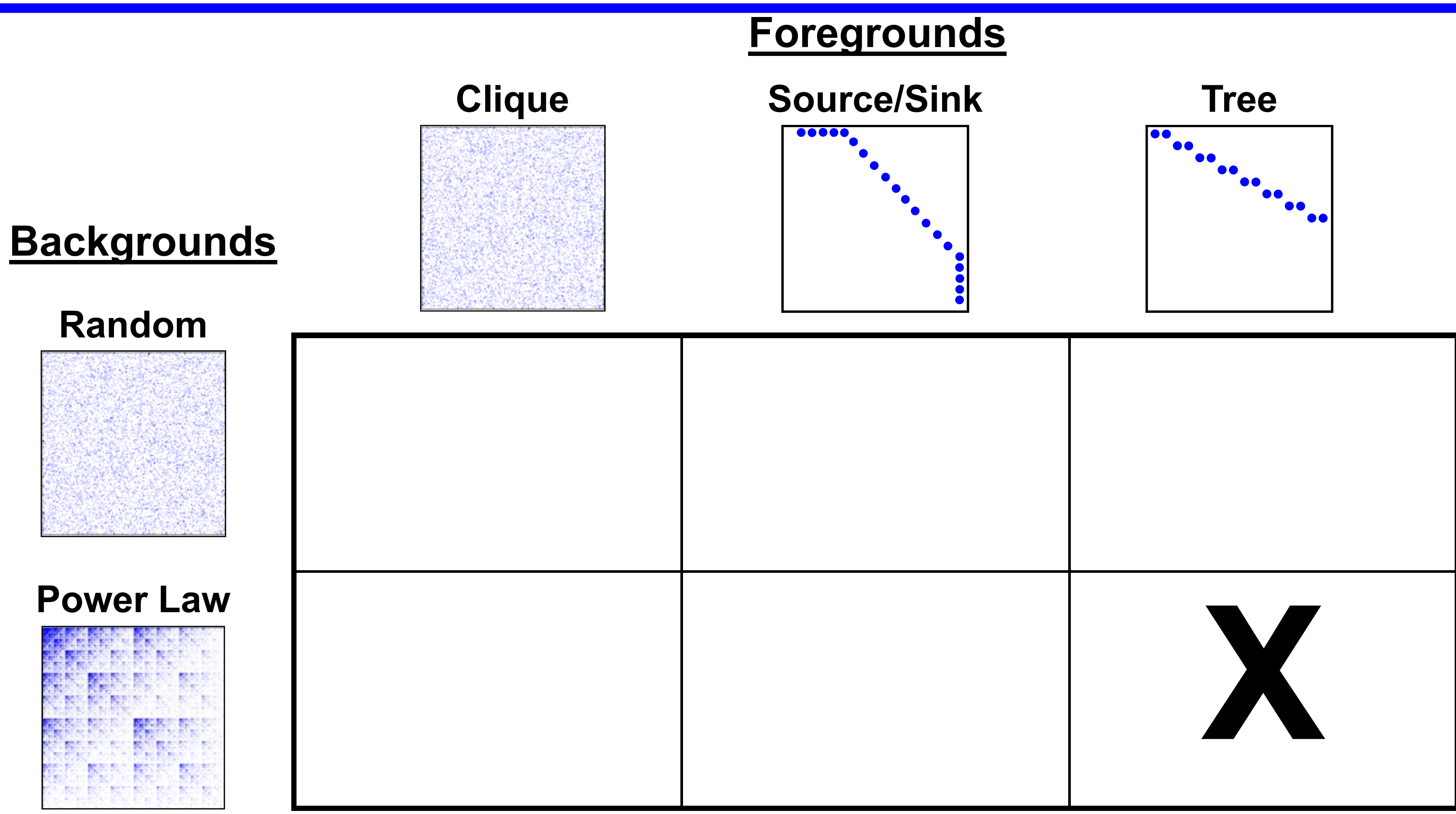
Algebraic Form

$$\mathbf{A} = \underset{n\times 1}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \otimes \underset{n\times n}{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}^{\otimes k-1} \otimes \underset{1\times n}{\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}}$$

Degree Distribution



Background/Foreground Combinations

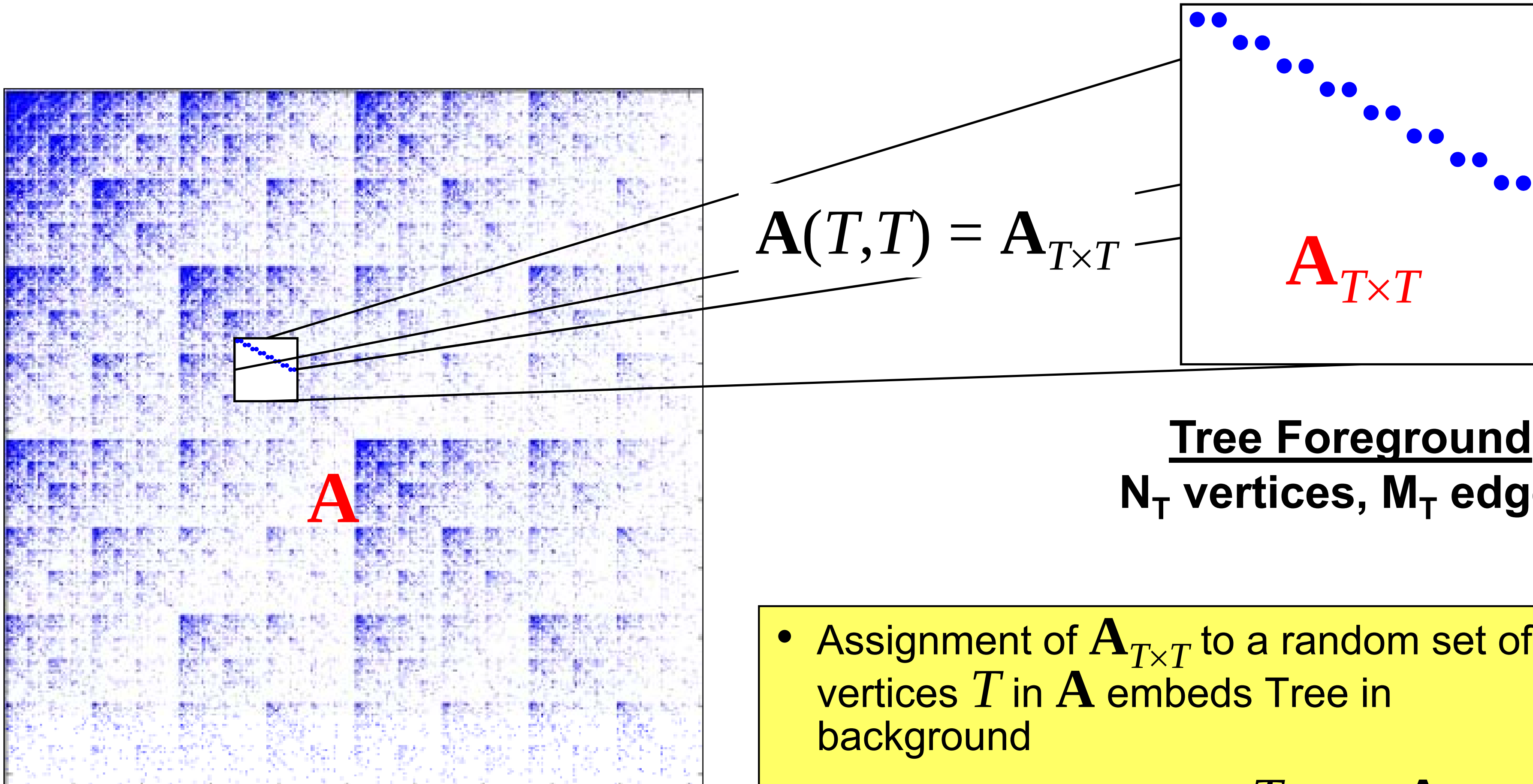


- Many interesting background/foreground combinations
- Rest of talk will focus on power law/tree

Outline

- Introduction
- Background and foregrounds
- **Tree Finding**
  - *Embedding*
  - *Cued vs Uncued*
  - *Set-Vector Representation*
  - *Algorithm*
  - *Results*
- Summary

Tree Embedding



Power Law Background
N vertices, M edges

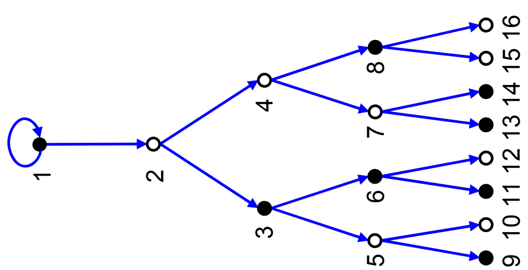
Tree Foreground
N_T vertices, M_T edges

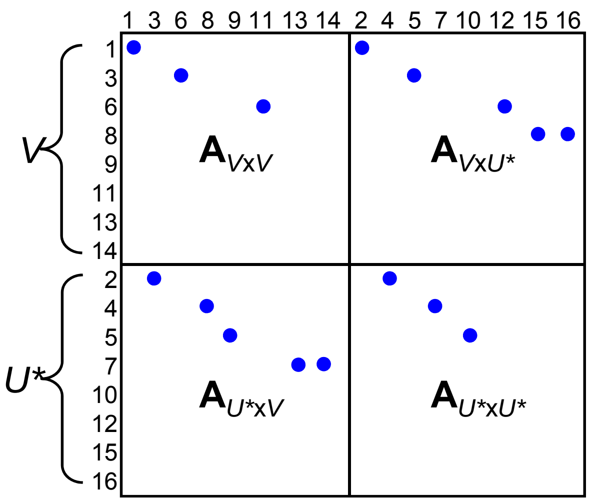
- Assignment of $\mathbf{A}_{T\times T}$ to a random set of vertices T in $\mathbf{A}$ embeds Tree in background
- Detection problem: find T given $\mathbf{A}$
  - Assume $N >> N_T$ and $M >> M_T$

Cued vs. Uncued Detection

- Uncued detection
  - No information about T is provided
  - Signal-to-noise ratio ~ N_T/N
  - Extremely difficult

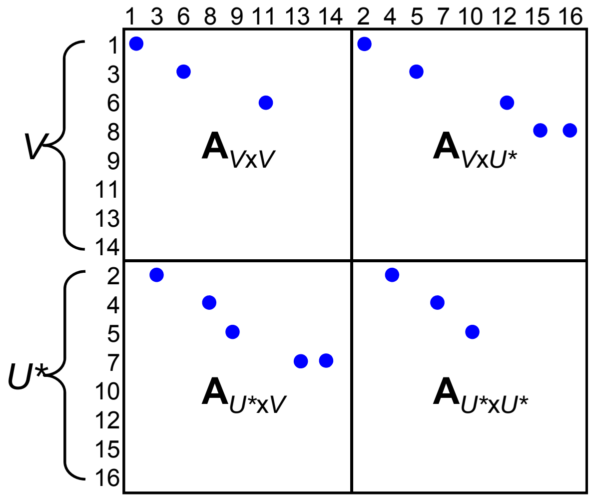
- Cued detection
  - T is divided into two sets V (given) and U^* (unknown)
  - More tractable



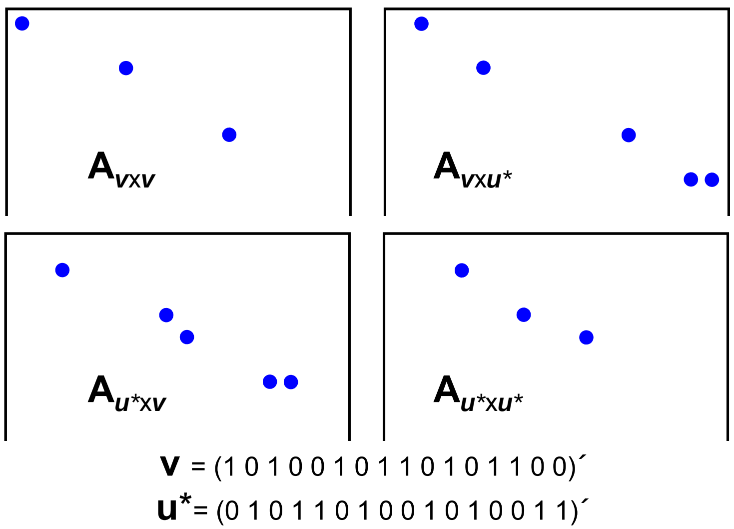


Set-Vector Dual Representation

Set Representation



Vector Representation



- Set of vertices V can also be represented as an N element vector where $\mathbf{v}(V) = 1$, allows multiple adjacency matrix representations

$$\mathbf{A}_{V \times V} = \mathbf{A}(V,V) \quad \text{or} \quad \mathbf{A}_{\mathbf{v}\times\mathbf{v}} = \mathbf{I}_{\mathbf{v}}\,\mathbf{A}\,\mathbf{I}_{\mathbf{v}}$$

- Set representation better for visualization
  – V contains only elements of interest
- Vector better for algorithm development and implementation
  – $\mathbf{v}$ allows linear algebraic transformations and preserves graph context

Tree Finding Algorithm Summary

- Step 0: Find all vertices that are 1st neighbors of $\mathbf{v}$

$$\mathbf{A}_{\mathbf{u}_0 \times \mathbf{v}} = \mathbf{A}\,\mathbf{I}_{\mathbf{v}} - \mathbf{A}_{\mathbf{v} \times \mathbf{v}}$$

$$\mathbf{d}_{\mathbf{u}_0 \times \mathbf{v}} = \mathbf{A}_{\mathbf{u}_0 \times \mathbf{v}} + \mathbf{A}'_{\mathbf{v} \times \mathbf{u}_0}$$

$$\mathbf{u}_0 = \mathbf{d}_{\mathbf{u}_0 \times \mathbf{v}} > 0$$

- Step 1a: Eliminate vertices that create too many connections to $\mathbf{v}$
- Step 1b: Eliminate vertices that connect to $\mathbf{v}$ that are filled
- Step 2: Find all vertices that are 1st neighbors of $\mathbf{v}$ that satisfy 1a & 1b
- Step 3: Select highest probability vertices based on (edges available) / (number candidates)
- Step 4: Select vertices with multiple connections into $\mathbf{v}$

Signal-to-Noise Estimate

- **Background power law: N = 2^{20}**
- **Foreground binary tree $N_T = 2^7$, f = 0.5 (fraction known)**

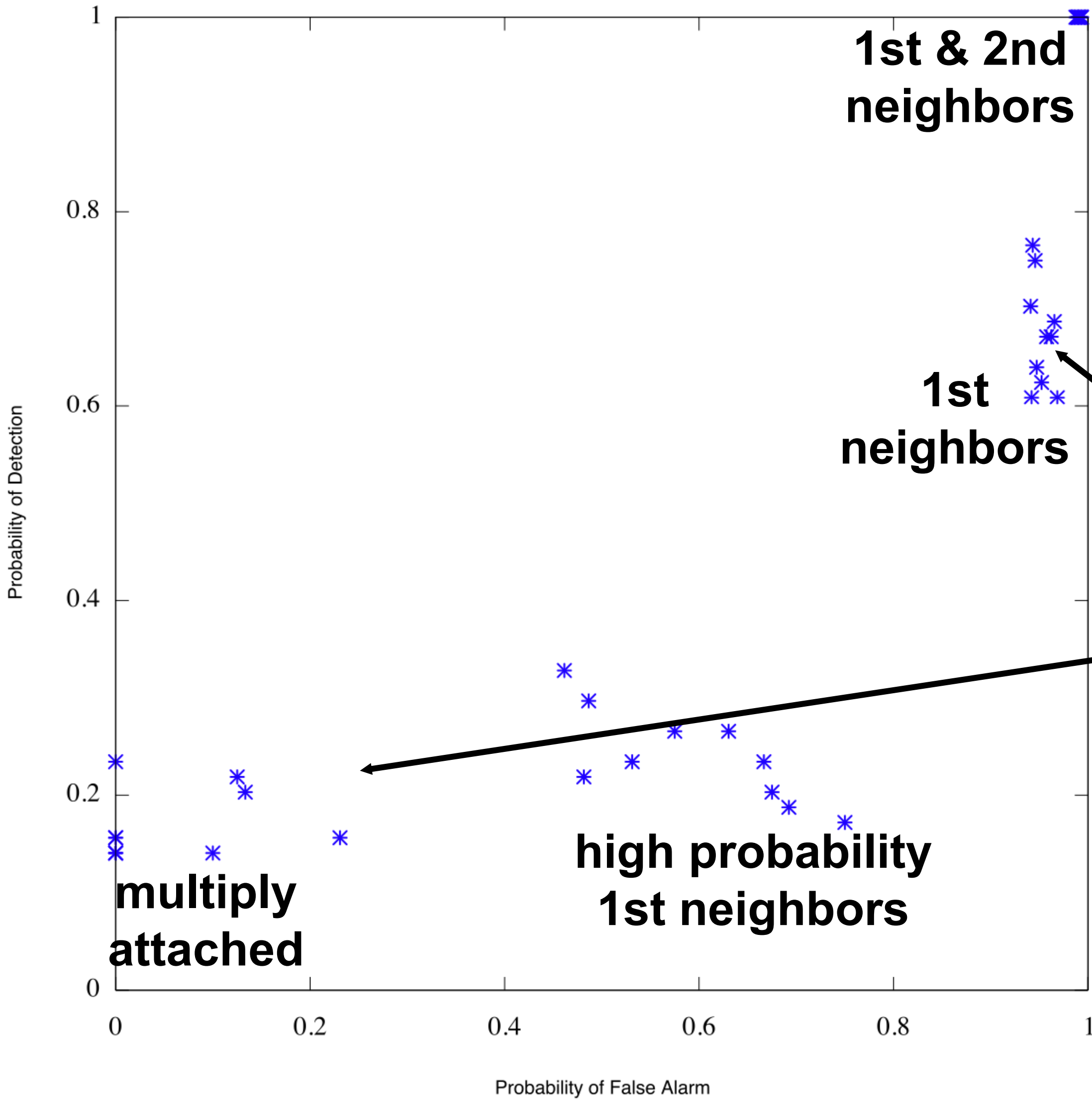
- **Baseline SNR ~ 2^{-14} ~ 0.00006**

- **1st and 2nd neighbors SNR ~ $5/2^{12}$ ~ 0.001**

- **1st neighbors SNR ~ $7/2^8$ ~ 0.03**
  - **Step 0**

- **Multiply attached neighbors SNR ~ 2^4 ~ 16**
  - **Step 4**

Probability of Detection (PD) vs Probability of False Alarm (PFA)



- PD vs PFA consistent with SNR estimates
- Can find most of tree with high PFA
- Can find subset of tree with very low PFA

Summary

- **Detection Theory**
  - Apply basic postulates of detection theory (signal, background, …)
  - Quantitatively estimate difficulty of problem (SNR)
  - Develop better detection algorithms

- **Linear Algebraic Graph algorithms**
  - Additional tools for algorithm development
  - Compact representation
  - Parallel implementation well understood