Are we there yet?
When to stop a Markov chain while generating random graphs

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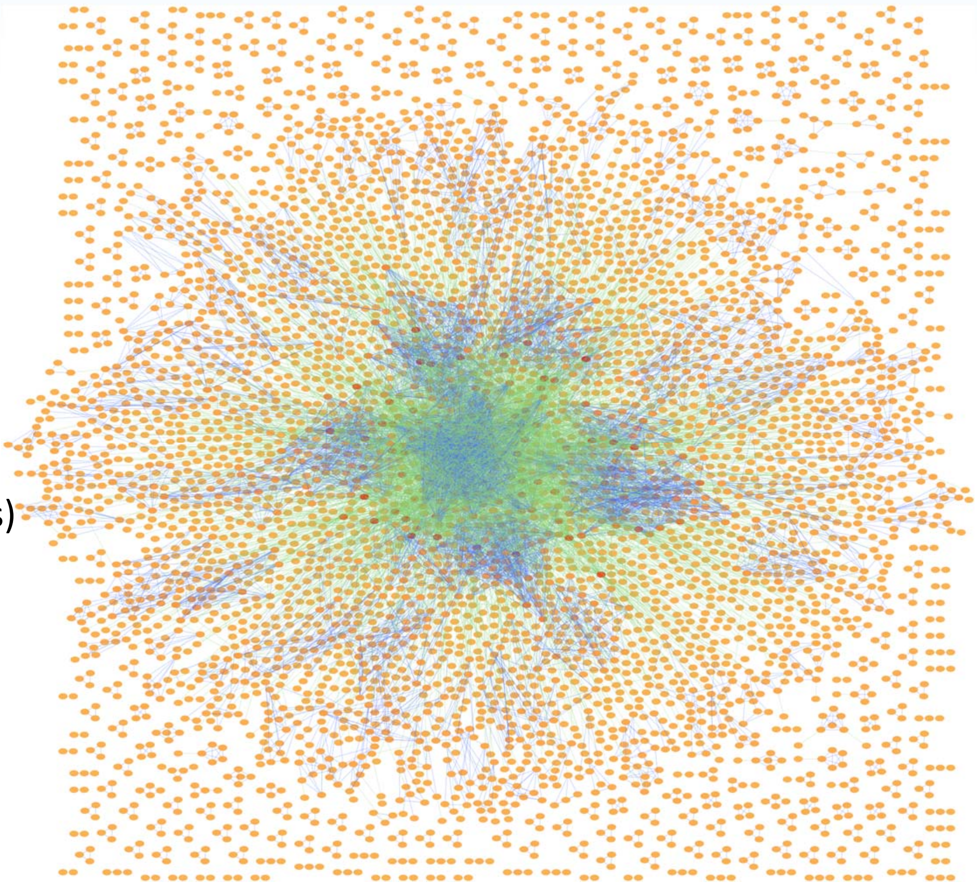
Why generate random graphs?

- Enable sharing of surrogate data
  - Computer network traffic
  - Social networks
  - Financial transactions
- Statistical analysis
  - Sample uniformly from a specified space
- Testing graph algorithms
  - Scalability
  - Versatility (e.g., vary degree distributions)
  - Characterizing algorithm performance
- Insight into…
  - Generative process
  - Community structure
  - Comparison
  - Evolution
  - Uncertainty



Block Two-Level Erdös-Rényi (BTER) graph; image courtesy of Nurcan Durak.

Markov Chains: common method to generate random graphs

- For this talk, a Markov chain (MC) is a graph whose nodes are realizations of a graph with desired features
  - Normally, MC graph is never constructed
  - We generate its vertices, as we walk on the graph
- A random walk on an MC (with the right features) can yield a random graph.
- To generate a random graph using an MC
  - Find an arbitrary node of the MC
  - Take a loooong random walk
  - You will arrive at a uniform random vertex of the graph
    - given that you have a ``good" MC
- Challenges
  - Generating a graph with given properties
  - Rewiring a graph to preserve desired features
  - Patience

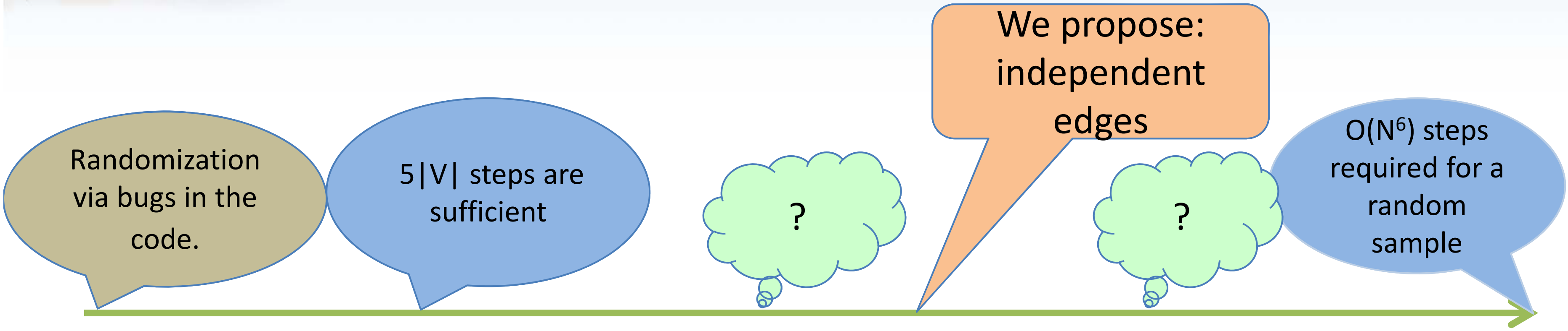
Math can prove convergence, but cannot grant you patience

Source: http://metsmerizedonline.com/wp-content/uploads/2013/02/Are-We-There-Yet.jpg

Can we find principled and practical metrics to think about convergence?

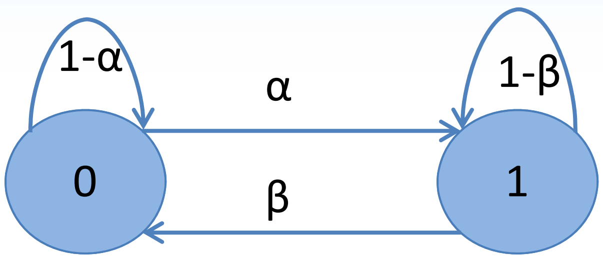
- In theory, we need to prove the MC eventually produces a random graph.
- In practice, bounds for convergence may be impractical or nonexistent.
- Practitioners use unprincipled methods.
  - e.g., 10K steps on the MC
- Interpretations of statistical tools may be hard.
  - What does Gelman Rubin test mean from a graphs perspective?

Can we find principled and practical metrics for convergence?



- What is a mathematically sound definition of “random enough?”
- Goals: practical, sound, and interpretable.
- An imperfect analogy:
  - To solve *Ax=b*, we do not compute $A^{-1}b$, we compute an *x*, that yields a small residual for *Ax-b*.
  - We learn how to deal with this imperfection.

Testing independence of edges



State 0: edge is absent

State 1: edge is present

α: probability that the edge will be inserted

β: probability that the edge will be deleted

$$T = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

T: transition matrix of the edge

- Assume the addition/deletion of an edge can be approximated as a Markov process.
- The full Markov chain (MC) can be approximated as a collection of smaller Markov chains.
- Convergence of the smaller MCs is a necessary condition for convergence of the full MC.

Convergence of smaller Markov chains

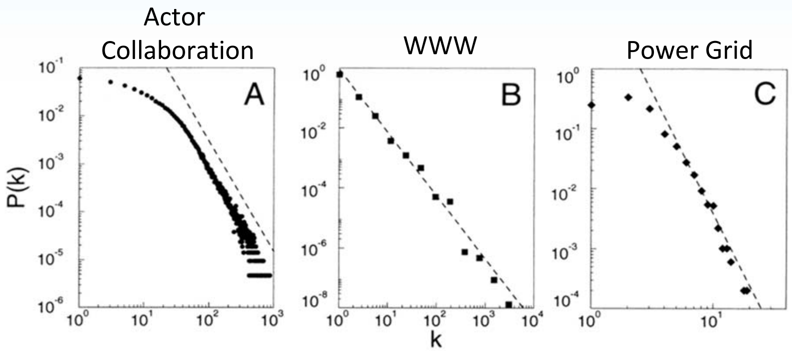
- Eigenvalues of *T* are *1* and $1 - (\alpha+\beta)$
- Eigenvalues form a basis, so initial state *v* can be written as $v=c_1e_1+c_2e_2$.
- After N iterations, we have

$$p = T^N v = c_1 e_1 + c_2 (1 - (\alpha + \beta))^N e_2$$

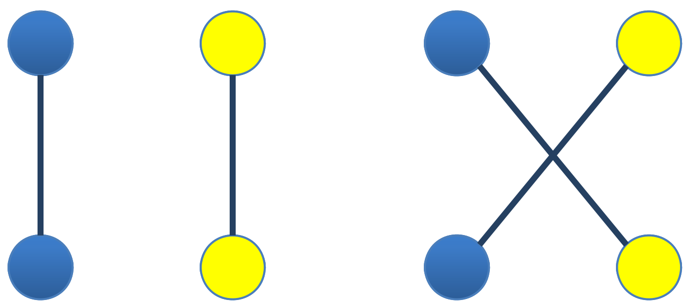
- The second term decays and p converges to c_1e_1, which indicates the probability the edge is present/absent in a random graph.
- For tolerance *ε,* the number of iterations required, *N*, is

$$N = \ln(1/\varepsilon)/(\alpha+\beta)$$

Preserving the degree distributions



A.-L. Barabasi and R. Albert. Emergence of scaling in random networks. Science, 286(5349):509-512, 1999.

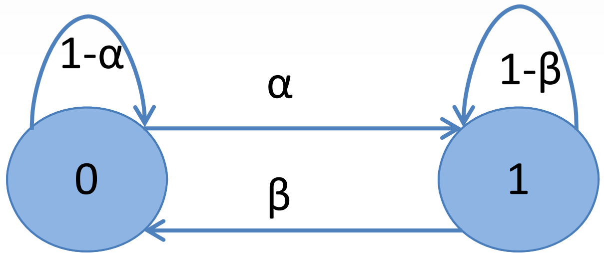


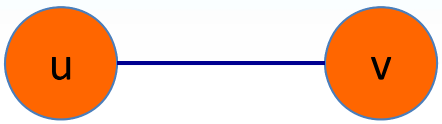
Choose two Random edges

Swap them

- Degree distribution is like a histogram of degrees.
- It is one of the critical features that distinguish real graphs from arbitrary sparse graphs.
- Rewiring scheme has long been used to perturb graphs while preserving the degree distribution.
  - Converges in $O(|E|^6)$-time.
- Havel and Hakimi described the first algorithm to construct a graph with a given degree distribution.

Transition matrix for preserving degree distribution





d_u: degree of vertex u

m: total number of edges

α: probability that the edge will be inserted

β: probability that the edge will be deleted

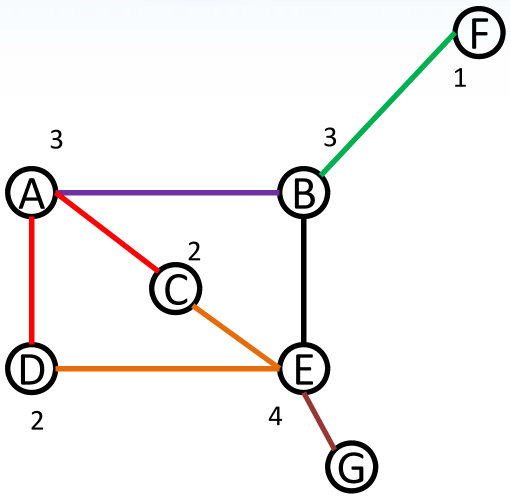
$$N = \ln(1/\varepsilon)/(\alpha+\beta)$$

$$\alpha = \frac{d_u d_v}{2m^2} \qquad \beta = 1-(1-\frac{1}{m})^2$$

$$\alpha+\beta \geq \frac{2}{m}$$

To generate a graph with independent edges with a specified degree distribution we need $N = \frac{m}{2}\ln\varepsilon^{-1}$

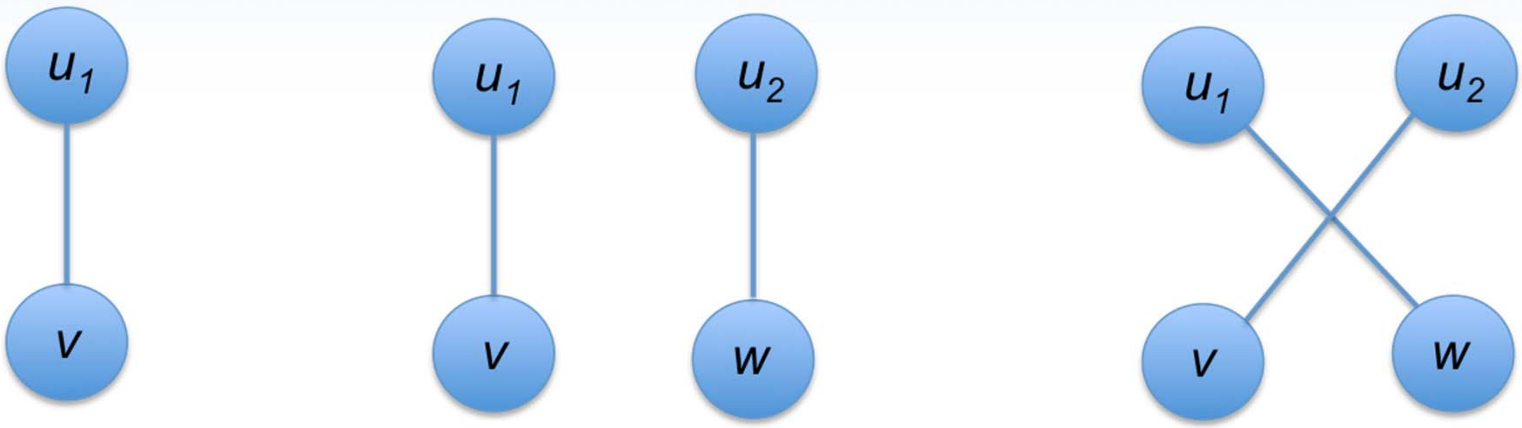
Joint Degree Distribution



Degree	1	2	3	4
1	0	0	1	1
2	.	0	2	2
3	.	.	1	1
4	.	.	.	0

- Joint Degree Distribution (JDD) specifies the number of *edges* between vertices of specified degrees.
- *JDD provides more information abot a graph.*
  - *The degree distribution is implicitly defined by JDD.*
- *Work on JDD is more recent and sparse.*

Preserving JDD



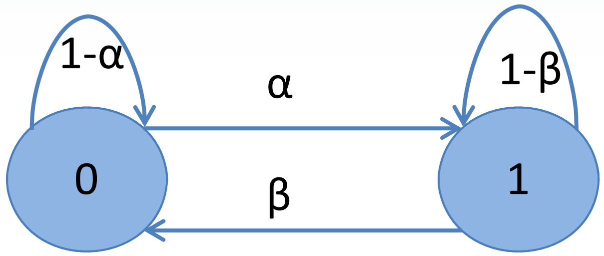
Step 1: Pick an edge (u_1,v), and pick one of its vertices, e.g., u_1

Step 2: Pick another edge (u,w), such that $d(u_1)=d(u_2)$ or $d(u_1)=d(w)$

Step 3: Swap edges

- This Markov chain can be used to construct uniformly random instances of a graph with a specified degree distribution.
- No theoretical bounds on convergence.
- A graph with a specified (feasible) joint degree distribution can be constructed in linear time.
- Stanton & P., *ACM J. Experimental Algorithmics, 2012*

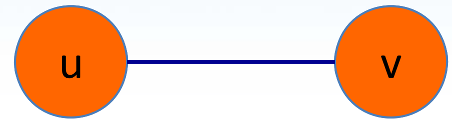
Transition matrix for preserving degree distribution



α: probability that the edge will be inserted

β: probability that the edge will be deleted

$$N = \ln(1/\varepsilon)/(\alpha+\beta)$$



d_u: degree of vertex u m: # edges

$f(d_u)$: #vertices of degree d_u

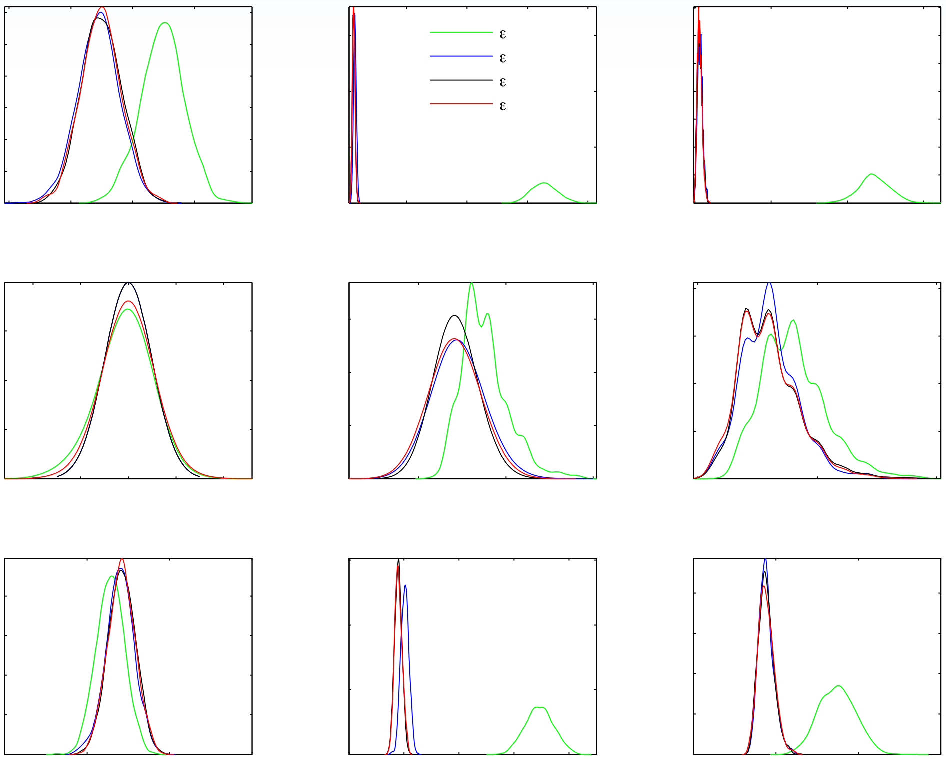
$J(d_u, d_v)$: #edges between d_u and d_v

$$\beta = \frac{1}{m} + \frac{f(d_u)-1}{2mf(d_u)} + \frac{f(d_v)-1}{2mf(d_v)}$$

$$\alpha \cong \frac{2J(d_u, d_v)}{mf(d_u)f(d_v)} \qquad \alpha+\beta \ge \frac{1}{m}$$

To generate a graph with independent edges with a specified degree distribution we need $N = m \ln \varepsilon^{-1}$

How does edge practice wok in practice?



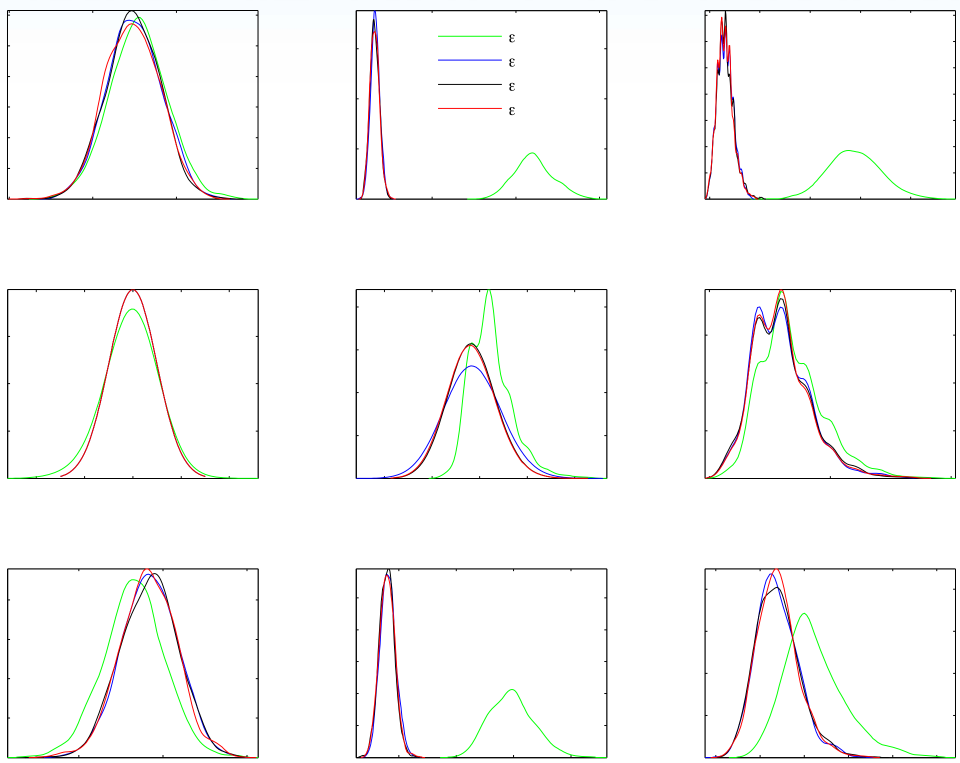
C. Elegans
297 vertices,
4296 edges

Netscience
1461 vertices,
5484 edges

Power
4941 vertices,
13188 edges

- Preserving degree distribution
- Errors correspond to 0.5|E|, 2.5|E|, 5|E|, and 7.5|E| iterations
- 1000 graphs generated starting from the original
- 5|E| iterations seem to be sufficient.

Edge independence in practice



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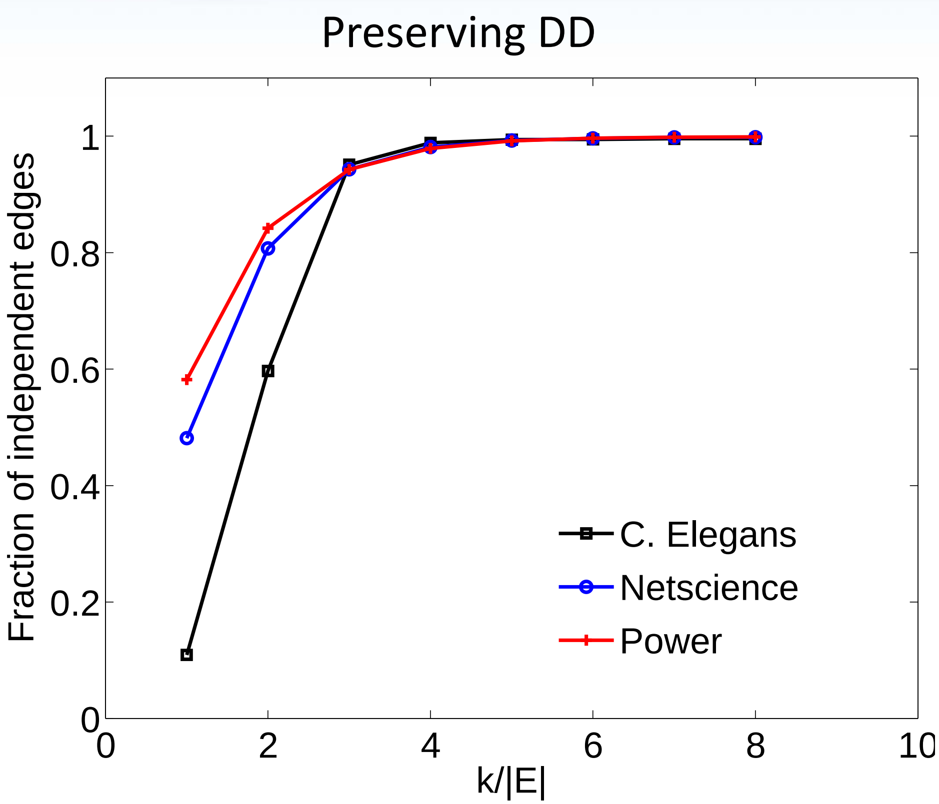
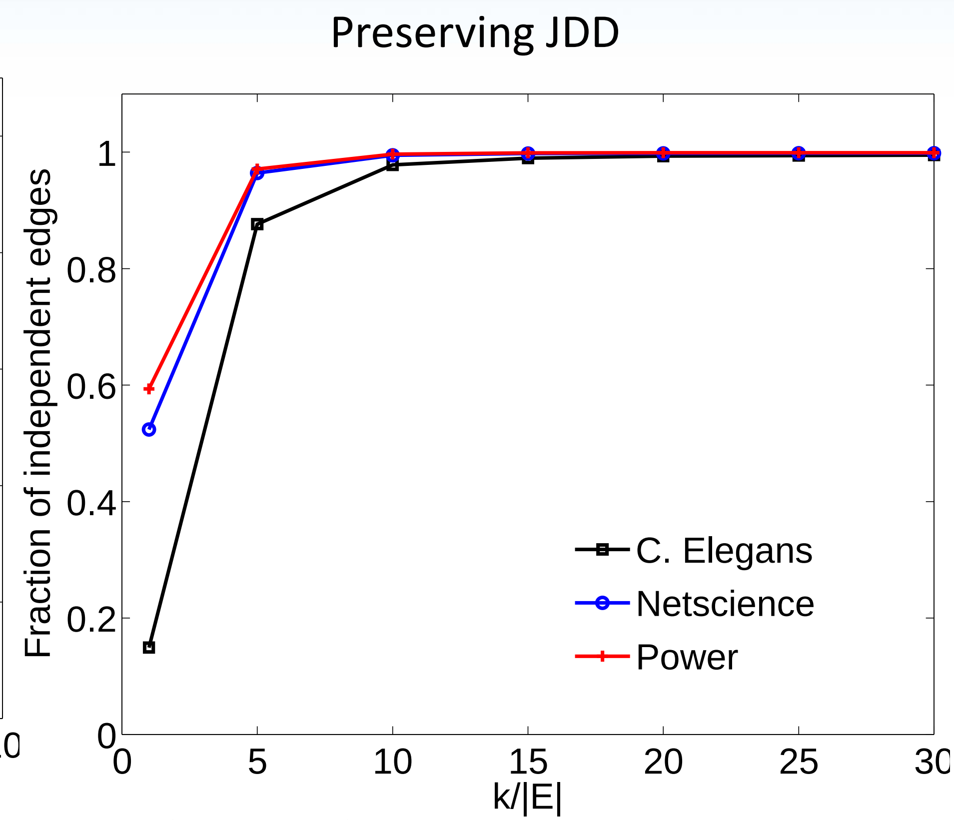
- Preserving JDD
- Errors correspond to |E|, 5|E|,10|E|, and 15|E| iterations.
- 1000 graphs generated starting from the original
- 10|E| iterations seem as sufficient.

Alternative way to measure independence

- Does knowing the current status (present/absent) of an edge help us predict its status in the next iteration better?
  - How about its status after 10 steps? 20 steps?
  - How many steps will be sufficient for the prediction to fail?
- The point we fail, the edge becomes independent
- A popular method in statistics
- Method:
  - Generate a long sequence
  - Fit a model to predict k steps ahead
  - Thin this sequence with smallest k for which the prediction fails

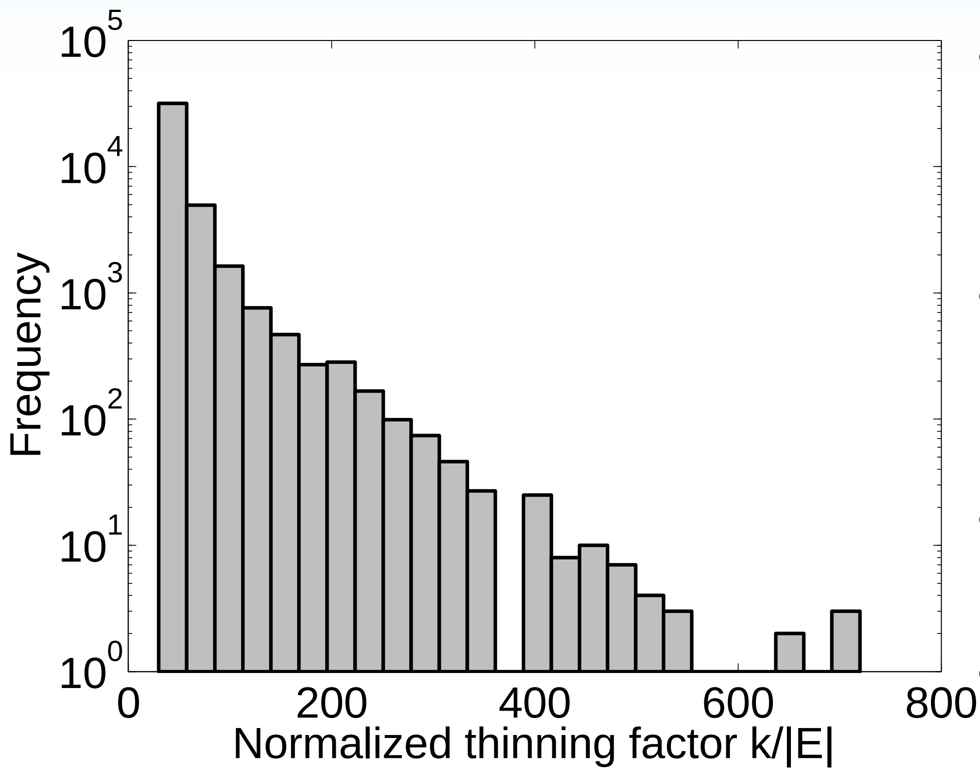
Edges become independent rapidly





- All potential edges are included in the analysis.
- Only a few remain after 7.5|E| and 15|E| iterations for preserving DD and JDD, respectively.

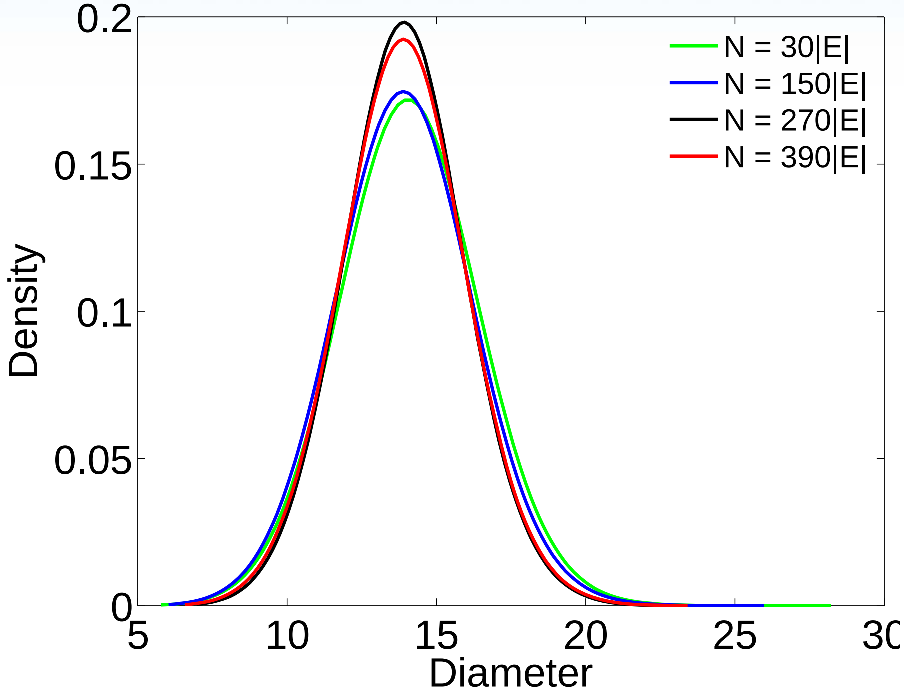
Some edges are tougher than others

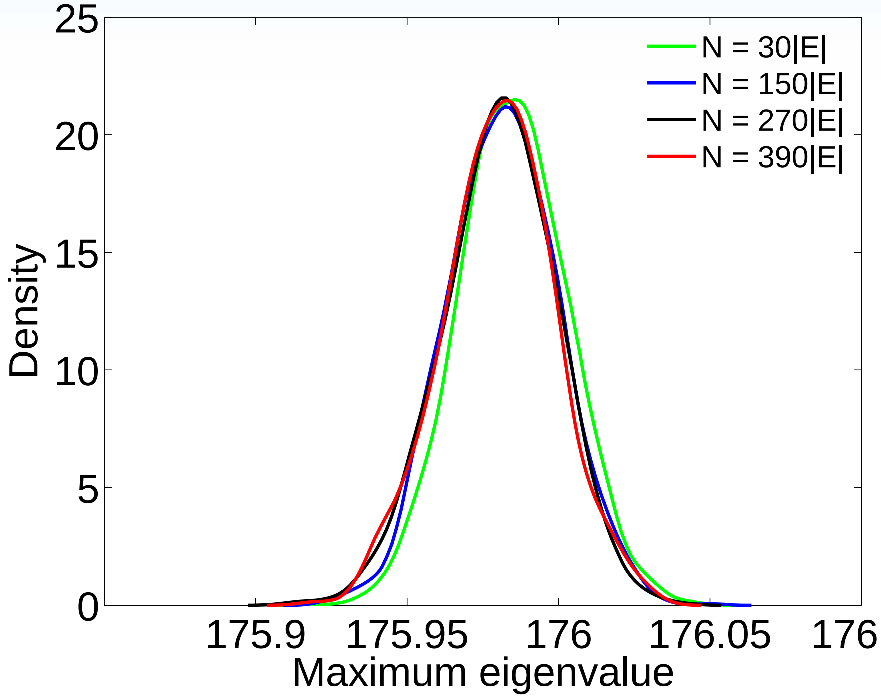


Soc-Epinions1
75879 vertices,405740 edges

- Preserving JDD on Soc-Epinions
  - Edges are sampled down to 10%.
- After 30|E| iterations 90% of the edges become independent.
- Most of the remaining ones are close to independence.
- There are a few outliers.

Diminishing returns for extra steps





- Preserving JDD on soc-epinions1
- Distributions are very similar

Conclusions

- Generating uniformly random instances of a graph with given properties is a fundamental problem in graph analysis.
- Markov chains are commonly used for this purpose, but guaranteeing/testing their convergence is a challenge.
- We proposed to use
  - edge independence as a practical metric.
  - smaller Markov chains for presence/absence of edges as a guide.
- We showed how the method applies to DD and JDD preserving MCs.
- Empirical studies on several graphs validated the approach.
- We are not guaranteeing convergence of the chain, but providing a metric that quantifies what is satisfied.
  - Results should be interpreted accordingly.
- The same approach can be used to guarantee independence of a bigger structures.

A new workshop

- SIAM Workshop on Network Science
- Dates: July 7-8, 2013
- Place: San Diego, CA
- Co-located with SIAM Annual Meeting
- There will be a call for posters
- Contact:
  - Ali Pinar (apinar@sandia.gov), Sandia National Labs
  - Madhav Marathe (mmarathe@vbi.vt.edu), Virginia Tech

Relevant Publications

- J. Ray, A. Pinar, and C. Seshadhri, ``A stopping criterion for Markov chains when generating independent random graphs," arXiv:1210.8184.
- J. Ray, A. Pinar, and C. Sehadhri, Are we there yet? When to stop a Markov chain while generating random graphs," Proc. WAW 12. .
- I. Stanton and A. Pinar, "Constructing and uniform sampling graphs with prescribed joint degree distribution using Markov Chains," ACM Journal on Experimental Algorithmics, Vol. 17, No. 1, 2012.
- I. Stanton and A. Pinar, "Sampling graphs with prescribed joint degree distribution using Markov Chains," ALENEX'11.