UPDATING MINIMUM WEIGHTED SPANNING TREES IN PARALLEL

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Minimum Weighted Spanning Tree (MST)

- Select a subset of edges from an undirected weighted graph (V,E), such that
  - (i) all the vertices are connected
  - (ii) the sum of the total edges is minimized

- Sequential Algorithms: Kruskal’s $O(E \log V)$ Prim’s $(E + V \log V)$
- Parallel Algorithms: Boruvka’s.

- Applications: Cluster Analysis, Circuit Design, Approximating TSP

- Our goal is to develop a parallel algorithm for updating MST as new edges are added and old edges are deleted.
Graph Sparsification

**Issue:** Massive size of the graphs makes it difficult to identify which portions to update

**Solution:** Use sparsification to identify only the edges that are important to the property under consideration

- **Key Edges:** Edges pertaining to the property (here edges in MST)
- **Remainder Edges:** Remaining edges (here everything but MST)
Insertion Operation

- Edge \((u,v)\) with weight \(W\) to be inserted
- Find path in MST from vertex \(u\) to vertex \(v\)
- Find the maximum weighted edge \((x,y)\) in the path \((wt=\text{max}W)\)
- If \((\text{max}W > W)\):
  - Add \((u,v)\) to MST; Delete \((x,y)\)
- Else:
  - Add \((u,v)\) to graph but not MST

Add edges \((A,F:1)\) and \((A,D:3)\)

Heaviest Edge in Path \(A\rightarrow F\) in MST is \(B-D:2\) or \(E-F:2\)
Replace either with \(A-F:1\)

Heaviest Edge in Path \(A\rightarrow D\) in MST is \(B-D:2\)
Do not replace \(A-D:3\)
Deletion Operation

- **Deletion Operation**
  - Delete Edge \((u,v)\) from the graph
  - Reconnect the tree (if possible) by finding minimum weighted edge connecting the two parts.

Deleted edge \((A,B:1)\) from MST
Deleted edge \((A,C:3)\) from remainder

Deleted edge \(A-B:1\)
Added \(C-F:2\) from remainder to rejoin the tree
Issues with Insertion-I

- Finding the path between \((u,v)\) for insertion—worst case complexity \(O(V+E)\)

- Complexity of simply re-doing the MST \(O(\text{ELogV})\)

- Therefore over multiple insertions time to update will be more than time to re-compute MST

- **Solution:** Store the paths (or maximum weighted edges) between vertex pairs. Requires \(O(V^2)\) storage
Finding Maximum Weighted Edges

- Find path from a designated root to all other vertices
- Mark the edges that have maximum weight in these paths
- Storage $O(V)$; Time $O(V+E)$
Finding Maximum Weighted Edges

**Case 1: (F:C)** Max Weight Edges are Different
Max Weight from F:D (E-F) 4
Max Weight from C:D (B-D) 3

Pick the highest weight edge (E-F) 4

**Case 2: (A:C)** Max Weight Edges are Same
Max Weight from A:D (B-D) 3
Max Weight from C:D (B-D) 3

Find path from A-C and then find max weighted edge B:C 2

If we keep track of the parent, the complexity of this at most $O(h)$; $h$=height of the tree
Selection of Root

- We need to select the root such that height of tree is minimized

- Assume we have no control over the original MST

- Find the longest path in the tree. Then select the vertex in the center of the path as the root

- Best Case $O(\log_k V)$; $k$=average branching
- Worst Case $O(V/2)$ but 50% chance that the max weighted edges will be different
Issues with Insertion-II

Inserting edges in parallel can lead to cycles

Case 1: Edges 1-5 and 1-6 are inserted in parallel.
Both find 3-4 as the edges to be deleted.
If added without synchronization creates a cycle.

Solution: Mark 3-4 with id of edge replacing it.
Only one id is possible.

Case 2: Edges 1-5 and 1-6 are inserted in parallel.
1-5 replaces 2-3 and 2-6 replaces 4-5.
If added without synchronization creates a cycle.

Solution: 2-3 and 4-5 must have the same weight.
Break ties by selecting edges with lower vertex ids.
Reduces to Case 1.
Issues with Deletion

• Deletion can be done in parallel by simply marking the edge as deleted.
• Finding edge to recombine the broken trees is expensive.
• May need to search all remainder edges $O(E)$.

Solution: Number of deletions in MST is less (4-5% of # of changes). Use Boruvka to update.
• Keep remainder edges in min-heap.
• Reduce the number of edges to search by using sparsification tree.
Sparsification Tree

- Introduced by Eppstein in 1997
  - Sparsification—a technique for speeding up dynamic graph algorithms by Eppstein et. al. JACM 1997
- Divided the edges of the graph into a binary tree
- Each node in the tree represents a subgraph
Sparsification Tree

• Sparsification tree reduce the number of edges we need to consider

• For Deletion \((u,v)\)
  • Only consider edges from the from the node where \((u,v)\) are in the same component to leaf
  • Delete (B-D): Search in nodes 5,1,2

• For Insertion \((u,v)\)
  • Only traverse the subgraph from the node where \((u,v)\) are in the same component to leaf
  • Insert (G-F): Traverse through subgraph at nodes 6,3,4
Putting It All Together

- **Input:** MST, Original Graph, Set of Changed Edges
- **Output:** Updated MST
- Create Sparsification Tree
- Place key edges and remainder edges in sparsification tree
- Select root of MST and find distances of all vertices
- Each changed edge is **processed in parallel**
  - **Insertion** \((u,v)\)
    - Find maximum weighted edge in path from \((u,v)\)
    - Replace as necessary
  - **Deletion** \((u,v)\)
    - Delete edge from MST
    - Rejoin by checking remainder edges
Experimental Setup

• Datasets
  • Experiments on random graphs created using RMAT
  • Vertices: $2^{18} - 2^{21}$
  • Edges: $8 \times$ Vertices

• Machine
  • Tusker at Holland Computing Centre
  • 6,784 cores interconnected with Mellanox QDR Infiniband along with 523TB of Lustre storage. Each compute node has 256 GB RAM and 4 Opteron 6272 (2.1 GHz) processors.
  • Shared memory implementation using OpenMP
Scalability Results
(repeated traversals-no rooted tree)

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<thead>
<tr>
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<th>Time in seconds</th>
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<tr>
<td>RMAT18</td>
<td>176.066872</td>
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<td>RMAT21</td>
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10K changes insertions and deletions mixed

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<td>RMAT21</td>
<td>2099.893274</td>
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</table>

5K changes insertions and deletions mixed

Time in seconds
Scalability Results (Rooted Tree)

Time to create rooted tree = 0.20 seconds (sequential)

Total time = .20 + time to update

<table>
<thead>
<tr>
<th>Rooted</th>
<th>Traversal</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>.41</td>
</tr>
<tr>
<td>4</td>
<td>.32</td>
</tr>
<tr>
<td>8</td>
<td>.27</td>
</tr>
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Conclusions

• The first parallel algorithm for updating MST
• Sparsification tree reduces amount of traversal
• Rooted tree method much faster than traversal—but need to make rooting parallel

• We can now handle updates for weighted trees.
• In lookout for other interesting properties to update

• Current code available at https://graphsparsification.herokuapp.com/
• Come to the Poster tomorrow